METRIC AND TOPOLOGICAL SPACES: EXAM 2026/27

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Problem 1 (5 + 5%)**.** Let $\big(\mathcal{X}, \mathrm{d}\big)$ be a non-empty metric space, r and s two positive radii, and $B^{\mathrm{d}}_r(x) = B^{\mathrm{d}}_s(y)$ for some $x, y \in \mathcal{X}$.

- Is it true that $r = s$?
- Is it true that $x = y$?

Problem 2 (15%)**.** A subset U of a space $\mathcal{X} \neq \varnothing$ is open if and only if for each point $x \in U$ there exists an open set $U_x \ni x$ such that $U_x \subseteq U$.

(prove)

Problem 3 (15%)**.** Let A and B be bounded subsets of a metric space $\big(\mathcal{X}, \mathrm{d}_{\mathcal{X}}\big)$ and $A \cap B \neq \varnothing$. Prove the inequality

$$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

(By definition, $\operatorname{diam}(\varnothing) = 0$ and $\operatorname{diam}(S) = \sup_{x,y \in S} \mathrm{d}_{\mathcal{X}}(x, y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Problem 4 (20%)**.** Let $\mathcal{X}$ be a space and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mathcal{Y}$ is connected and $\mathcal{Y} \subseteq \mathcal{Z} \subseteq \overline{\mathcal{Y}}$, then $\mathcal{Z}$ is connected.

(prove)

Problem 5 (20%)**.** Let A and B be compact subsets of a Hausdorff space $\mathcal{X}$. Prove that the intersection $A \cap B$ is compact in $\mathcal{X}$.

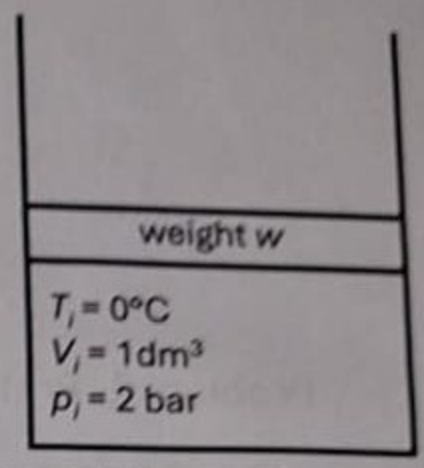
Problem 6 (20%)**.** Let $\mathcal{X} = [a, b] \sqcup [c, d]$ be the disjoint union of two segments in $\mathbb{E}^1$, and suppose $f, g \colon \mathcal{X} \to \mathcal{X}$ are Banach contractions. Does there always exist a point $x_0 \in \mathcal{X}$ such that $g\big(f(x_0)\big) = x_0$, and if yes, how many such x_0 can there be ?

(state and prove)

Date: November 3, 2026. Good luck !

Problem 3. Enthalpy & Entropy [9 pts]

A cylinder is filled with a monoatomic ideal gas ($T_i = 0°C$, $p_i = 2$ bar, $V_i = 1$ dm^3), and is closed by a frictionless mass of weight w. The cylinder is placed in a large lecture theater ($T_0 = 20°C$, $p_0 = 1$ bar, $V = 100$ m^3).



a) The gas in the cylinder heats up isobarically to 20°C. What is the final volume of the gas in the cylinder? [2 pts]

b) What is the amount of heat absorbed by the gas as it heats up to 20°C? [4 pts]

c) For this new situation (after the gas has heated up), it turns out there is a small hole in the cylinder through which the gas slowly leaks out. What is the change in entropy after all gas has leaked out and mixed into the lecture theater? Assume T_0 remains at 20°C. [3 pts]

Constants:

Avogadro's number:	$N_A = 6.02\ 10^{23}\ mol^{-1}$
Boltzmann constant:	$k_B = 1.381\ 10^{-23}\ J/K$
Gas constant:	$R = 8.31\ JK^{-1}mol^{-1}$
Atomic mass unit (u):	$m_u = 1.67\ 10^{-27}\ kg$
Electronvolt:	$1\ eV = 1.6022\ 10^{-19}\ J$

Please check whether your name and student number are on every sheet

Thermal Physics I, test 2

30 Oct 2026, 18.15-20.15

Write your name and student number on every sheet

Extra-time students: 10 minutes per hour => 20 min extra

Problem 1. Preliminaries [4 pts]

a) Provide a short yet accurate statement of the Third Law of Thermodynamics. [2 pts]

b) Derive the Maxwell relation [2 pts]

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

Problem 2. The Stirling Cycle [12 pts]

A Stirling engine uses 1 mole of an ideal monatomic gas as its working substance. The cycle consists of the following reversible processes:

$1 \rightarrow 2$: Isothermal expansion at $T_H = 600$ K from volume V_1 to $V_2 = 3\,V_1$, absorbing heat Q_{12}.

$2 \rightarrow 3$: Isochoric cooling from T_H to $T_L = 300$ K, rejecting heat Q_{23}.

$3 \rightarrow 4$: Isothermal compression at T_L from V_2 back to V_1, rejecting heat Q_{34}.

$4 \rightarrow 1$: Isochoric heating from T_L back to T_H, absorbing heat Q_{41}.

The gas constant is $R = 8.314$ J mol^{-1}K^{-1}.

a) Sketch the Stirling cycle on a pV diagram, clearly labeling all four processes, the four heat flows, and indicating the direction of the cycle. [4 pts]

b) Calculate the heat absorbed during the four steps of the cycle. [4 pts]

c) Determine the net work done by the gas over one complete cycle. [2 pts]

d) What is the theoretical efficiency of this Stirling cycle? How does this compare to the efficiency of a Carnot cycle operating between the same two temperatures? Explain briefly. [2 pts]